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MTH 512
HW #7

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Question #1

$m_A(x) = x^3 \quad \& \quad IN(E_0) = 3$

$C_A(x) = x^7$

Hence $A_{7\times 7}$ has only two possible Jordan Forms:

$$J = J_0^{(3)} \oplus J_0^{(3)} \oplus J_0^{(1)}$$

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$J = J_0^{(3)} \oplus J_0^{(2)} \oplus J_0^{(2)}$$

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Question #2

let A be symmetric matrix ($A = A^T$) , then define

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^n \quad \underline{\underline{\text{s.t.}}} \quad \forall\, Q = (a_1, \dots, a_n) \in \mathbb{R}^n$$

$$T(Q) = A Q^T$$

Now, $\langle T(Q), Q \rangle = \langle A Q^T, Q \rangle$

$$= (A Q^T)^T \cdot Q^T$$

$$= Q A^T \cdot Q^T$$

$$= \langle Q, A^T Q^T \rangle$$

$$= \langle Q, T^*(Q) \rangle$$

Hence, by inner product property $\Rightarrow \langle T(Q), Q \rangle = \langle T^*(Q), Q \rangle$ (*)

$$\Rightarrow T(Q) = T^*(Q)$$

$$\Rightarrow A Q^T = A^T Q^T$$

$\Rightarrow T$ is symmetric.

We know that : $T(Q) = \alpha Q$, for $Q \neq 0$

$$\langle T(Q), Q \rangle = \langle \alpha Q, Q \rangle = \alpha \langle Q, Q \rangle$$

(*) $\langle Q, T^*(Q) \rangle = \langle Q, T(Q) \rangle = \langle Q, \alpha Q \rangle = \bar{\alpha} \langle Q, Q \rangle$

$\Rightarrow \alpha = \bar{\alpha} \quad \Rightarrow \alpha$ is a real number.

Question ~~#~~3

let A be orthogonal matrix $(A^T = A^{-1})$, then define

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^n \quad \underline{\text{s.t.}} \quad \forall Q = (a_1, \dots, a_n) \in \mathbb{R}^n$$

$$T(Q) = AQ^T$$

Now,
$$\begin{aligned} \langle T(Q), Q \rangle &= \langle AQ^T, Q \rangle \\ &= (AQ^T)^T \cdot Q^T \\ &= Q A^T Q^T \\ &= \langle Q, A^T Q^T \rangle \\ &= \langle Q, A^{-1} Q^T \rangle \\ &= \langle Q, T^*(Q) \rangle \end{aligned}$$

Hence, $T^*(Q) = A^{-1} Q^T = T^{-1}(Q)$

$\Rightarrow$ T is orthogonal

By : $T(Q) = \alpha Q$, $Q \neq 0$

$$\langle T(Q), Q \rangle = \langle \alpha Q, Q \rangle = \alpha \langle Q, Q \rangle$$

$$\langle Q, T^*(Q) \rangle = \langle Q, T^{-1}(Q) \rangle = \langle Q, \tfrac{1}{\alpha} Q \rangle = \tfrac{1}{\bar{\alpha}} \langle Q, Q \rangle$$

$$\Rightarrow \alpha = \frac{1}{\bar{\alpha}} \quad \Rightarrow \alpha \bar{\alpha} = 1$$

$$\Rightarrow |\alpha| = 1$$

Question #4

- let $A_{n\times n}$ be invertible matrix s.t. $A^T = A$
- let $B = A^2$

Define $T : \mathbb{R}^n \to \mathbb{R}^n$ s.t. $\forall Q \in \mathbb{R}^n$, $T(Q) = AQ^T$

Define $F : \mathbb{R}^n \to \mathbb{R}^n$ s.t. $\forall P \in \mathbb{R}^n$, $F(P) = A^2 P^T$
$= B P^T$

$\Rightarrow$ Want to show that $B^T = B$:

$$B^T = (A^2)^T = (AA)^T = A^T A \quad (\text{since } A^T = A)$$
$$= A^T A^T$$
$$= AA = A^2 = B$$

$\Rightarrow$ want to show that All eigenvalues of B are real :

(Same method as Q#2)

$$\langle F(P), P \rangle = \langle BP^T, P \rangle$$
$$= (BP^T)^T P^T$$
$$= P B^T P^T$$
$$= \langle P, B^T P^T \rangle$$
$$= \langle P, F^*(P) \rangle$$

Hence $F^*(P) = F(P)$

From Q#2 $\Rightarrow$ all eigenvalues of B are real.

$\Rightarrow$ want to show that each eigenvalue is > 0 :

$$\langle F(P), P \rangle = \langle T^2(P), P \rangle$$
$$= \langle T(P), T^*(P) \rangle$$
$$= \langle T(P), T(P) \rangle \quad \neq 0$$
$$= \| T(P) \|^2 > 0$$

Since $|A| \neq 0 \quad \Rightarrow \quad \alpha = 0$ is not eigenvalue .

$\Rightarrow \quad \langle F(P), P \rangle > 0 \quad \& \; F = F^*$

$\Rightarrow$ F is positive definite .

Question № 5

$$J = J_2^{(3)} \oplus J_2^{(1)} \oplus J_2^{(1)} \oplus J_6^{(3)} \oplus J_6^{(3)}$$

① $n = 11$

② $m_A(x) = (x-2)^3 (x-6)^3$

③ $C_A(x) = (x-2)^5 (x-6)^6$

④ $IN(E_2(A)) = 3$

⑤ $IN(E_6(A)) = 2$

A is not diagonizable since we have repeated roots.

Question № 6

$C_A(x) = (x-3)^3 (x+4)^2$

$m_A(x) = (x-3)(x+4)^2$

$\Rightarrow J = J_3^{(1)} \oplus J_3^{(1)} \oplus J_3^{(1)} \oplus J_{-4}^{(2)}$

$\Rightarrow IN(E_3) = 3$ & $IN(E_{-4}) = 1$

$$J = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix}$$

Question #7

let $A_{n\times n}$ be symmetric s.t. $A^T = A$, & Assume non-zero points $Q_1, Q_2 \in \mathbb{R}^n$ & Some real numbers $a \neq b$

s.t.
$$A Q_1^T = a Q_1^T$$
$$A Q_2^T = b Q_2^T$$

$\Rightarrow$ Show that Q_1 & Q_2 are orthogonal.

$\boxed{\text{Case 1}}$ If $\langle Q_1, Q_2 \rangle = 0$, done

$\boxed{\text{Case 2}}$ If $\langle Q_1, Q_2 \rangle \neq 0$

$$\begin{aligned} a \langle Q_1, Q_2 \rangle &= \langle a Q_1, Q_2 \rangle \\ &= \langle T(Q_1), Q_2 \rangle \\ &= \langle Q_1, T^*(Q_2) \rangle \\ &= \langle Q_1, T(Q_2) \rangle \\ &= \langle Q_1, b Q_2 \rangle \\ &= \bar{b} \langle Q_1, Q_2 \rangle = b \langle Q_1, Q_2 \rangle \end{aligned}$$

$\Rightarrow$ Hence $a = b$!! contradiction

since we assume $a \neq b$

$\Rightarrow$ Thus Q_1, Q_2 orthogonal.

Question #8

$$c_A(x) = m_A(\alpha) = (\alpha-1)^4(\alpha+5)^5$$

$\Rightarrow$ $A_{9\times 9}$ matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 \end{bmatrix} = J$$

$\Rightarrow$ $IN(E_1) = 1$

$IN(E_{-5}) = 1$